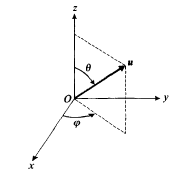
Homework 10 for General Physics II-Answer Key

By SJ

In the class, I derived the expression for the important states and operators of a spin1/2 system using results of SG experiments. In the |+Z>-|-Z> basis:

 ,

And for arbitrary direction u: 



All the problems below, the matrices are all shown as in |+Z>,|-Z> basis

1. Express the |+Z> and |-Z> state in terms of combination of |+x>-|-x> as well as combination of |+y>-|-y>.

Answer:



The 4 coefficients of each case form a matrix. (i.e. |z> to |x> transform matrix from the above calculation is:  and . These matrices are the inverse for the transform of  (from |x> to |z>) and  respectively, but they are just the transposed and complex-conjugate of the original too! i.e . The transform with this property is called Unitary Transform. (Unitary Transform is the transform that will not alter the module of the ‘vector’, i.e. ， you know all these a while ago in mechanics)

1.  For a SG setup shown above, two magnetic fields arranged along z and x direction. The experimental results show two electron spots, corresponding to  of the spin along the x-direction. N electrons (N is large) come out of oven, half will pass and half will be blocked after SGz; What are the number of electrons appear in the final detection screen for each spot? (We may add one more SGz after the SGx and let both beams passing through the last SGz, what are the experimental observation?

Answer:

From the statement in the problem we already know that N/2 electrons will be in the Sz+ beam. For these N/2 electrons, after SGx setup, part of it will end in Sx+ and part will be in Sx-. The probability of each is same as finding the eigenvalues of Sx.



So there will be total of N/4 in Sx+ beam and N/4 in Sx- beam.

(If we insert another SGz after the SGx, and let both beams pass SGz. Then Sx+, Sx- beams each will split into two beams corresponding to the Sz+ and Sz-. So on a X-Z screen of detection, there will be 4 spots, each with N/8 on average.

1. If we prepare a spin state of electron in forms of , now we carry out SG experiment with this input state. 1) Measure the Sz, what are the values of measurement? What are the probability for each result? What is the average of this measurement?(expectation value) 2) Answer same questions for the Sx measurement.

Answer:

1. The only possible Sz measurement results are eigenvalues, so either . The probability of detecting each is given by:



The average of the measurement is given by expectation value:

Of course, this may be calculated more easily by:



1. For the Sx measurement, results are also either . The probability of getting each is:



The average is:



Or 

The above calculation using matrix is straightforward. However, I will rework the problem in a slight different way, which is the origin for the matrix method and bears more physical meaning.

1. For the Sz measurement, it is natural to express the state as a linear combination of the eigenstates which are |+z>, |-z> in this case. The expression provided in the problem already satisfies this since we use |+z>.|-z> as basis:



The c’s are probability amplitude of detecting eigenvalues associated with |+z>,|-z> eigenstates. The probability is just their module squared.



The expectation value is easily calculated with the definition of eigenstates:



1. For the Sx case, it is natural to express the input state as linear combination of |+x>,|-x> (eigenstates). Since we already know the relation between the |+z>,|-z> with |+x>,|-x> (see problem 1), then the expression of input as linear combinations of |+x>,|-x> can be worked out. Then the rest would follow the same procedure. (I shall skipped the details here)
2. For an arbitrary direction of magnetic field in the SG experiment, say it is along direction u as shown in the figure above. The expression for Su is also provided above. Find the eigenvalues and associated eigenstates of Su. ( show that  and )

Answer:

This is a problem of finding eigenvalues and eigenvectors knowing the matrix, the procedure is standard:

, let be eigenvalues and |+u>,|-u> be associated eigenstates. From the definition of eigenvalues and states:

, in the |+z>,|-z> basis,, the a, b need to be determined.

Finding eigenvalues in 2-dimension is simple:



This is what you would expect, since the space is isotropic.

The determination of |+u>:



, so  within a phase factor. Because this phase factor would be common to both *a* and b, so we can safely neglect it.

So the |+u> is same as the form provided in the problem(commondoes not matter), similarly you can find the |-u>.

1. For a series of SG experiment, the first SG is along z and acting as a ‘polarizer’, i.e. prepare the spin state into |+Z>, then this |+Z> will pass the second SGu which is along the direction u as in last problem. What are the experiment results and the probability? How about we prepare a state in |+u> first and then let it pass through Sx measurement?

Answer:

With the last problem worked out, this problem is straightforward:

1. Input state |+z>, Su measurement only get either  (the electrons will split along direction u of course):



1. |+u> as input, pass Sx, also only get either  (the electrons will split along direction x in this case) :

For the other probability, you probably guess that it would be 1/2-. That would be correct.



1. The idea is to find matrix elements for A, and B (and their eigenvalue will be determined too in the due process, knowing the 2-D matrix, finding eigenvalue is standard procedure as worked out in detail in problem 4).

The A, B are observables in 2-D, so the requirement of Hermitian would reduce the unknowns, A, B have to be in forms of , so only 3 unknowns to be determined (they can be complex numbers).

From the measurement result, we already know the 11 element (the *a*) for both A,B:

1. For A, *a*=1/2, there is only one condition left and that seems impossible to get all b and c, which is right. You may make matrix multiplication and use last result to get info. Or like what I did here by expansion:

 , insert completeness of |1>,|2> between A’s:



(Remember  is just A12 element1 of matrix A), so b=0. This is all I can know about this observable A provided with measurement infos. It has one eigenvalue 1/2, and another one c is unknown.

1. For B, *a*=1 from measurement, similarly either using product of matrices or like what I did using insertion of completeness:





The matrix of B has one unknown phase angle then:

, it is not diagonal as A, but eigenvalues can be computed easily, from Det(B-)=0 or use the fact the sum of eigenvalues equals trace, and products of eigenvalues equals Det. ( I shall skip the steps), the answers should be: 1/2 and 3/2

1. （Optional） From the given relations (it is actually called anti-commutator), we do some “cosmetics”:  (1)

And  (2)

If we make right product of  on formula (2), we get:



1. Use the above relation:



(I used Tr(AB)=Tr(BA) in the second equation)

1. Let the eigenvalues of  be 



(Here you can put  will give same result)

Since the trace is zero, means  so the two eigenvalues have to be 1 and -1 each.

1. Starting from definition of linear independence:

 (4) for linear independence, need to prove the relation only holds for c’s all equal 0. These are matrix so that one equation above means many equations for each elements, that seems too complicated. Here the trick is to use provided relations between matrix and property of trace proved above:

Let’s right and left product of  on relation (4):



Add those two and use relations, 

 (5)

Take the trace for the above:



But we have proved =0, and so cj=0; and then from (5) c0=0 too (the are not Null matrices from the condition given), so starting from (4), we derived the c’s have to be all 0, that means linear independence for the 4-matrices.

8. Prove that the position operator X in momentum representation (1-D):



Answer: There are a few methods for the proof. We know the X,P operator in space representation, and the eigenstates of |x>, |p> are:



Method 1: The X acts on  if expressed in space would just be:



The space representations of any wave vectors are related to momentum representation by Fourier Transform:





The new wave function in momentum space after X act on state, would be a Fourier transform of , i.e.





QED.

Method 2: (details neglected)

Insert identity into the formula:

 this is because we know what the X acts on state in space representation; but since we need to find its action in momentum space, we need further expand in terms of :

We know the expression of every term and also the property of delta function (in this proof we need the exponential expansion form of delta function and also use the property of delta function:, easy to prove using integration by parts), then we can derive same results as method 1.

9. In the experiment shown below, we first select |+z> state after electron passing first SGz device (M-field gradient along Z, the |-z> were blocked); then let it pass second SGn whose M-field is along direction ; then we select electron only in |+n> state; let them pass the 3rd SGz and we select |-z> electrons. The second SGn has direction along 

SGz



SGn



SGz



1. What fraction (the same meaning as probability) of particles that transmitted by the first SG will pass the last SG?
2. What is the angle for the 2nd SGn so to maximize the particles that will pass the last SG?
3. If the 2nd SGn is removed, what will be the fraction of particles that pass the last SG?

Answer:

1. Obviously everything would be easy using the |+z>, |-z> basis, and the |+n> expression is: (for )



Then the fraction of particles in |+z> state after 1st SG that will pass the 2nd is:



The fraction of particles in |+n> after 2nd SG that will pass the 3rd is:



So the total probability for the sequence is:



1. 

Clearly  (The SGn is along x direction, becomes SGx) will maximize pass percentage. P=1/4 in this case.

Here this is really like polarized light, let H polarizer select H polarized light, then pass a +45 polarizer and then pass a V polarizer. This will give maximum light output (We use Malus law there for polarized light). The similarity is due to same math (polarized light in Jones vector form is exactly same as spin state matrix form).

1. If the 2nd SG being removed, no particle can pass 3rd SG, since <-z|+z>=0 (or use polarized light analogy)

10. The state of a spin 1/2 particle is given by:



We measure Sz for particles in such state. A) What is the average (expectation value) of <Sz>, and what is the standard deviation . Note the  is defined as: 

B) What is the time evolution of the state if particle initially in above state is subject to a constant B field along z direction and interaction Hamiltonian is given by: 

Answer:

A) The expectation value can be calculated with:



Or using:



For the :



(here we use result: If A|a>=a|a>, then f(A)|a>=f(a)|a>)

From definition of :



B) Since [Sz, H]=0 (the operators commute), the eigenstate for Sz is also eigenstate for H, i.e.

, with eigenvalues of .

Then as derived in notes:



The value of uncertainty on Sz is independent of time, since it commutes with H, so the  as in A). (of course you may carry out calculation to verify this, but from property of good quantum operator, we can draw this conclusion quickly)

11. Study of an operator. In this problem we shall investigate an operator which may look strange (or unfamiliar) at beginning, but through a systematic study, we will start building a feeling for its physical significance. The operator is defined as:



 is an angle parameter and Sz is the spin component along z. It may appear clueless at present what this operator will do and we will see below:

1. What is R’s effect on the |+z>, |-z>?
2. What is R’s matrix form in |+z>,|-z> basis?
3. Is R unitary, i.e. 
4. What is R’s effect on other |+x>, for simplicity, we set  here to see this operation will change |+x> state into what state?
5. From above, can you see what this R operator does? (hint: rotation)
6. Puzzle: when, there is some peculiarity. Prove  for spin 1/2 particles.

Comment: It is weird (counter-intuitive) because rotation by  will not restore state but with a minus sign! This is clearly due to spin quantum number may not be integers (here is 1/2) and is a quantum effect. Though a total phase factor for a state will not affect measurement on such state, but in an interference experiment, where state with difference phase factor will show difference, and thus using such interference experiment, people can experimentally verify this quantum peculiarity for spin 1/2 particles (ref. Werner et al. *Phys. Rev. Lett.* **35**, 1053 (1975))

Answer:

1. We use result: If A|a>=a|a>, then f(A)|a>=f(a)|a> (or you expand the function of operators into Taylor series, and that is how the above is proved)

Since:  then:



Clearly R will not change state (i.e. |+z>, |-z> are eigenvectors and eigenvalues are a phase factor)

1. The matrix representation is clear from above action on base vectors (Since z’s are eigenvector, and the matrix will be diagonal with eigenvalues. Of course you may apply other methods I told you to determine the form). In matrix form:



1. The adjoint form of R is:

, the R is clearly not Hermitian. But easily to see that



So R is unitary.

1. For 



Take a careful look on the right side, we see that it is just |+y> state with a common phase factor , and thus we conclude the operation changes spin 1/2 particle from |+x> to |+y>

1. With above simple calculation, we may strongly guess it is a rotation along z axis by angle  (as usual convention, counter clock-wise, c.c.w)! You see that it does not change the |z> state (imagine that you rotate SGz along z. of course this will not affect except for a phase factor). When you rotate |+x> state along z by 90 degree will change it into |+y> (rotate a SGx by 90 degree will certainly change it into SGy). I hope this can convince you that Rz is the rotation operator. Of course you may try other angles to see further that this is the case (say 45 degree, and |+x> will change into a |+45>… or 90 degree, but act on |+y>, it will change into |-x> as a c.c.w rotation along z will do)
2. Indeed there is a puzzle for the rotation on spin1/2 (or non-integer) state. As  a full rotation along z, the phase factor will not be 1 for spin1/2.



For arbitrary spin 1/2 state, we can express it in |+z>, |-z> basis:



Such change of phase factor can only be tested in interference experiment as the reference did. (directly measurement of the state will not show the effect of common phase factor, remember only the relative phase is physically important)